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The effective long-time diffusivity for a passive scalar in a Gaussian model fluid flow

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The problem considered is the diffusion of a passive scalar in a 'fluid' in random motion when the fluid velocity field is Gaussian and statistically homogeneous, isotropic and stationary. A self-consistent expansion for the effective long-time diffusivity is obtained and the approximations derived from this series by retaining up to three terms are explicitly calculated for simple idealized forms of the velocity correlation function for which numerical simulations are available for comparison for zero molecular diffusivity. The dependence of the effective diffusivity on the molecular diffusivity is determined within this idealization. The results support Saffman's contention that the molecular and turbulent diffusion processes interfere destructively, in the sense that the total effective diffusivity about a fixed point is less than that which would be obtained if the two diffusion processes acted independently.

1. Introduction

In his fundamental paper 'Diffusion by continuous movements' (1921) Taylor pointed out that the long-time diffusion of a passive scalar field in a fluid in homogeneous, isotropic and stationary random motion should be described by the simple diffusion equation. His argument may be summarized as follows. The displacement of a typical particle is the vector sum of its displacements in the subintervals into which the time interval may be divided. If the subintervals are large in number and each of a duration long compared with the correlation time or eddy circulation time then the displacements in different subintervals are 'almost' independent and it is likely that their sum will have a Gaussian distribution. Although a rigorous derivation of this result has not yet been given there is little doubt that it is essentially correct. Computer simulations of diffusion in a given Gaussian velocity field (Kraichnan 1970*a*) confirm Taylor's predictions except in the case of two-dimensional time-independent flow. Taylor's analysis leads also to an expression for the effective long-time diffusivity in terms of the integrated Lagrangian velocity correlation function. The problem of relating the Lagrangian to the Eulerian velocity correlation function has received much attention and several approximation procedures have been proposed which lead to good agreement with Kraichnan's computer simulations in many cases (Kraichnan 1970*a*,*b*; Phythian 1975; Lundgren & Pointin 1976). The effective long-time diffusivity may therefore be calculated with reasonable accuracy in such models.

The generalization of Taylor's work to the case in which the molecular diffusivity is non-zero has been carried out by Saffman (1960). An interesting question which arises concerns the way in which the molecular and turbulent diffusivities combine to give an effective value. Taylor suggested that the two were simply additive, Townsend argued that 'constructive interference' would occur (Batchelor & Townsend 1956), while Saffman has given strong reasons for believing that 'destructive interference' takes place (Saffman 1960; see also Batchelor, Howells & Townsend 1959). Unfortunately the simple approximations mentioned above for the case of zero molecular diffusivity are difficult to generalize to non-zero values of this parameter. More complicated theories exist. For example the direct-interaction and Lagrangian-history direct-interaction approximations developed for the full turbulence problem may also be formulated for the Gaussian flow model considered here. No detailed calculations have been carried out for non-zero molecular diffusivity but a qualitative argument based on these theories supports Saffman's view (Kraichnan 1965). [The term direct interaction will be used in this paper to denote the formulation of that approximation for the diffusion of a passive scalar by a fluid whose velocity field has prescribed statistics.]

In the present paper we should like to explore an alternative approach to the calculation of effective diffusivity. The method has something in common with direct interaction in that it is derived from a perturbation expansion for the 'propagator' (the averaged Green's function of the equation of the scalar field). However, whereas direct interaction follows from the truncation of a series in which the full propagator is determined self-consistently (this term will be explained below), the series considered here involves a self-consistency requirement only on the propagator's asymptotic form for large time differences. Since Taylor's argument tells us that this asymptotic form is determined by a single parameter, the effective diffusivity, the procedure leads to a self-consistent expansion for this quantity. Such series, to which the terms 'consolidated' and 'renormalized' are also applied, have been used extensively in many branches of physics. (For a discussion see Martin, Siggia & Rose 1973.) They have the property that the unknown quantity, in this case the effective diffusivity, is expressed as an infinite series of terms which themselves involve the unknown quantity. The self-consistency condition is comparatively simple for the series derived here and higher-order terms can be found by evaluating Feynman-type integrals.

We have found that if a suitable special choice of the velocity-field correlation function is made it is possible to evaluate terms up to sixth order in the perturbation series without undue difficulty. Successive approximations for the effective diffusivity, as a function of the molecular diffusivity κ, obtained in this way appear to be converging and are consistent with Kraichnan's exact values for $\kappa = 0$. The results are given for time-dependent and 'frozen' (i.e. time-independent) velocity fields, and clearly show that destructive interference of turbulent and molecular diffusion takes place. Rather strangely, it appears that the convergence is more rapid for the frozen-field case, which is usually regarded as the most stringent test of approximation schemes. The method is exact for the two trivially soluble cases of uniform velocity fields and velocity fields with delta-function time correlations.

The introduction of renormalized perturbation series into turbulence theory has proved very fruitful (for a recent review see Kraichnan 1975). However very little is known about the convergence or asymptotic nature of such series. Rigorous convergence proofs seem to be lacking and the numerical calculation of terms beyond the lowest order poses severe computational problems. Investigation has therefore been largely confined to examining such series for simple models such as the anharmonic oscillator (Morton & Corrsin 1970) and convection by a uniform velocity field (Kraichnan 1964). The work reported here extends these ideas to less simple models (although they are still simple by comparison with real fluid flow) and may therefore help to shed some further light on this approach.

2. Renormalized series for effective diffusivity

The basic equation of the problem may be written

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\mathscr{G}(\mathbf{x},t;\mathbf{x}',t') = \delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')-U_\alpha(\mathbf{x},t)\frac{\partial}{\partial x_\alpha}\mathscr{G}(\mathbf{x},t;\mathbf{x}',t'), \tag{1}$$

where κ is the molecular diffusivity and $\mathbf{U}(\mathbf{x},t)$ is the Eulerian velocity field of a realization of the random flow. The quantity $\mathscr{G}(\mathbf{x},t;\mathbf{x}',t')$, which is required to vanish for $t < t'$, is the causal Green's function of the problem. It enables one to calculate, for a particular realization of $\mathbf{U}$, the scalar field arising from a source distribution $s(\mathbf{x},t)$ by means of the expression $\int dx' \int dt' \mathscr{G}(\mathbf{x},t;\mathbf{x}',t')\,s(\mathbf{x}',t')$. It should be noted that $\mathscr{G}$ is a random function because of its dependence on $\mathbf{U}$.

The random velocity field $\mathbf{U}(\mathbf{x},t)$ is assumed to be solenoidal, Gaussian and statistically homogeneous, isotropic and stationary with a correlation function given by

$$\langle U_\alpha(\mathbf{x},t)\,U_\beta(\mathbf{x}',t')\rangle = R_{\alpha\beta}(\mathbf{x}-\mathbf{x}',t-t').$$

It is often convenient to work with the Fourier transform

$$\tilde{R}_{\alpha\beta}(\mathbf{k},t) = \int d\mathbf{x}\,R_{\alpha\beta}(\mathbf{x},t)\,e^{-i\mathbf{k}.\mathbf{x}},$$

which, because of the incompressibility and isotropy conditions, has the form

$$\tilde{R}_{\alpha\beta}(\mathbf{k},t) = (\delta_{\alpha\beta}-k_\alpha k_\beta/k^2)\,\tilde{R}(k,t).$$

The average of $\mathscr{G}$ over all realizations of the flow is denoted by G and will be referred to as the propagator. Because of the homogeneous and stationary nature of the probability distribution of $\mathbf{U}$ it is seen that G is a function of $\mathbf{x}-\mathbf{x}'$ and $t-t'$:

$$G(\mathbf{x}-\mathbf{x}',t-t') = \langle\mathscr{G}(\mathbf{x},t;\mathbf{x}',t')\rangle.$$

It is clear that G enables one to calculate the mean value of the scalar field, at any position and time, arising from a non-random source function s by means of the integral $\int dx' \int dt' G(\mathbf{x}-\mathbf{x}',t-t')\,s(\mathbf{x}',t')$. Thus G describes what is usually referred to as the single-point diffusion problem. Multiple-point diffusion, which may be described by averages of products of $\mathscr{G}$ functions, will not be considered here.

A formal series solution for $\mathscr{G}$ may be generated from (1) by treating the term $(\mathbf{U}.\nabla)\mathscr{G}$ as a perturbation. Averaging over $\mathbf{U}$ then gives a series for G. However such a 'bare' perturbation expansion is of little value and the terms may not even be well defined if κ is zero. We shall therefore consider instead a renormalized series. A rough argument which describes the basic philosophy behind this approach proceeds as follows. Since Taylor's work shows that the average effect of the convection term

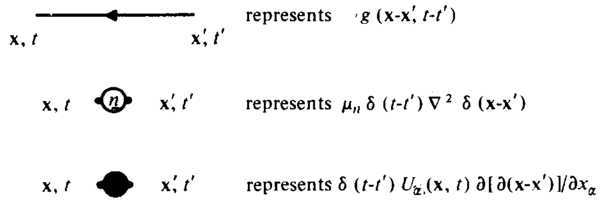

FIGURE 1

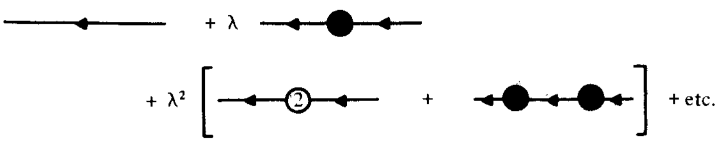

FIGURE 2

$(\mathbf{U}\,.\,\nabla)\mathscr{G}$ is to increase the effective diffusivity, it seems sensible to replace κ by a larger value and to subtract compensating terms from $(\mathbf{U}\,.\,\nabla)\mathscr{G}$, treating the difference as a perturbation. Since this new perturbation consists of the convection term with some of its effect subtracted out it will be smaller, in some sense, and hopefully the corresponding perturbation series will converge more rapidly than the bare series.

The renormalized series may be obtained by a rearrangement of the bare series but for our purpose it is more convenient to adopt a slightly different approach. We consider an equation containing a parameter λ:

$$(\partial/\partial t-\mu\nabla^2)\,\mathscr{G}(\mathbf{x},t;\mathbf{x}',t') = \delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')+(\lambda^2\mu_2+\lambda^4\mu_4+\ldots)\,\nabla^2\mathscr{G}(\mathbf{x},t;\mathbf{x}',t') + \lambda U_\alpha(\mathbf{x},t)\,\partial\mathscr{G}(\mathbf{x},t;\mathbf{x}',t')/\partial x_\alpha, \tag{2}$$

where

$$\mu+\mu_2+\mu_4+\ldots = \kappa \tag{3}$$

and the μ_n are, at this stage, arbitrary. For $\lambda = -1$ this equation reduces to (1) while for $\lambda = 0$ it becomes

$$(\partial/\partial t-\mu\nabla^2)\,\mathscr{G}(\mathbf{x},t;\mathbf{x}',t') = \delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'),$$

the causal solution of which will be denoted by $g(\mathbf{x}-\mathbf{x}',t-t')$. Rewriting (2) as an integral equation gives

$$\mathscr{G}(\mathbf{x},t;\mathbf{x}',t') = g(\mathbf{x}-\mathbf{x}',t-t')+\int d\mathbf{y}\int d\tau\, g(\mathbf{x}-\mathbf{y},t-\tau)\,(\lambda^2\mu_2+\lambda^4\mu_4+\ldots) \times \nabla_y^2\,\mathscr{G}(\mathbf{y},\tau;\mathbf{x}',t')+\lambda\int d\mathbf{y}\int d\tau\, g(\mathbf{x}-\mathbf{y},t-\tau)\,U_\alpha(\mathbf{y},\tau)\frac{\partial}{\partial y_\alpha}\mathscr{G}(\mathbf{y},\tau;\mathbf{x}',t'), \tag{4}$$

from which $\mathscr{G}$ may be obtained as a power series in λ. As usual the terms of the series are most conveniently represented by diagrams. If we adopt the definitions given in figure 1 then $\mathscr{G}(\mathbf{x},t;\mathbf{x}',t')$ is represented by the series shown in figure 2. The line ending on the left of each diagram is understood to be at the space–time point $(\mathbf{x},t)$ and that on the right at $(\mathbf{x}',t')$. An integration over all intermediate positions and times is implied.

Taking the average over U and making use of its Gaussian property gives for $G(\mathbf{x}-\mathbf{x}',t-t')$ the series shown in figure 3, where a broken line indicates that the two U's connected by it are replaced by the corresponding correlation function (see, for example, Phythian 1972). A quantity Σ is defined by the sum of all the diagrams

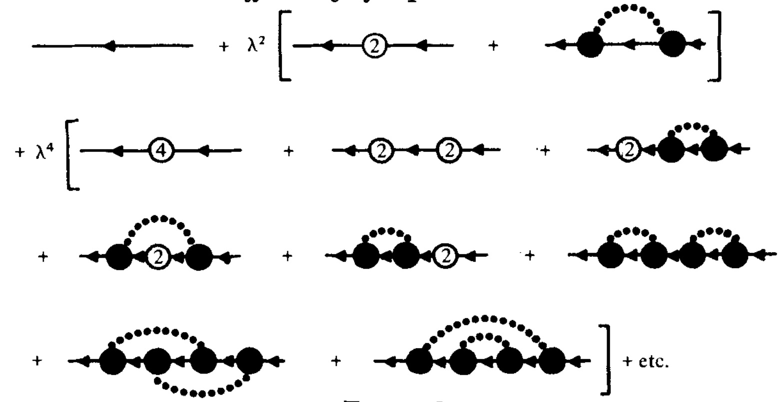

FIGURE 3

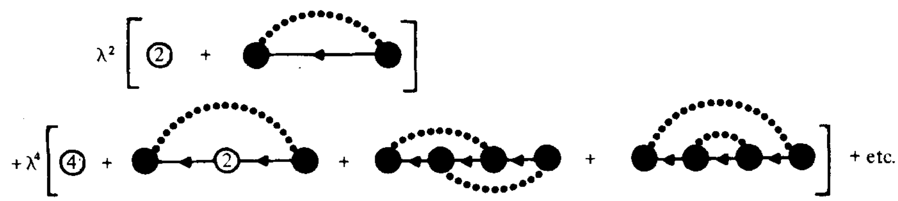

FIGURE 4

which cannot be separated into two disconnected parts by severing a single internal g line. Σ is analogous to the self-energy in quantum theory but a more appropriate term in the present context would be the generalized diffusivity (for a discussion see Kraichnan 1970*b*). $\Sigma(\mathbf{x}-\mathbf{x}', t-t')$ is then given by the series shown in figure 4 and G is related to Σ by the equation

$$G(\mathbf{x}-\mathbf{x}',t-t') = g(\mathbf{x}-\mathbf{x}',t-t') + \int d\mathbf{y} \int d\tau \int d\mathbf{y}' \int d\tau' g(\mathbf{x}-\mathbf{y},t-\tau) \times \Sigma(\mathbf{y}-\mathbf{y}',\tau-\tau')\, G(\mathbf{y}'-\mathbf{x}',\tau'-t'). \tag{5}$$

This may be rewritten in terms of Fourier transforms as

$$(\partial/\partial t + \mu k^2)\, \tilde{G}(\mathbf{k},t-t') = \delta(t-t') + \int d\tau\, \tilde{\Sigma}(\mathbf{k},t-\tau)\, \tilde{G}(\mathbf{k},\tau-t'). \tag{6}$$

If Taylor's argument is correct then, for time intervals large compared with the eddy circulation time and wavenumbers small compared with the inverse length scale of the turbulence, we have

$$\tilde{G}(\mathbf{k},t-t') \approx \exp\{-\sigma k^2(t-t')\},$$

where the constant σ describes the large-scale diffusion, relative to a fixed point, due to the combined effect of molecular and turbulent motion.

Considering (6) for large $t-t'$ and small k, using the above asymptotic form for $\tilde{G}$, and assuming that $\tilde{\Sigma}$ falls off sufficiently rapidly for large time differences for the following integral to exist (which can be verified at each order of the perturbation series), we find for small k

$$(\mu-\sigma)\, k^2 = \int_{0-}^{\infty} d\tau\, \tilde{\Sigma}(\mathbf{k},\tau) \exp(\sigma k^2 \tau).$$

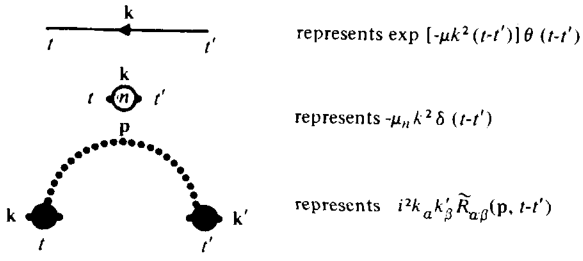

FIGURE 5

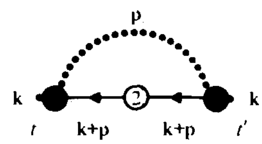

FIGURE 6

However $\tilde{\Sigma}(\mathbf{k},\tau)$ is of order k^2 for small k (again this follows from its diagram structure) so we have finally

$$\mu-\sigma = \text{coefficient of } k^2 \text{ in} \int_{0-}^{\infty} d\tau\,\tilde{\Sigma}(k,\tau). \tag{7}$$

The lower limit of the integral has to be taken as $0-$ to avoid any ambiguity in the terms involving $\delta(\tau)$.

Using the series in figure 4 for Σ in (7) gives an expansion for $\mu-\sigma$. The terms are most easily evaluated in terms of the Fourier-transformed diagrams. Each line now carries a wavenumber, and the total wavenumber is conserved at each vertex (see figure 5). For example, the diagram shown in figure 6 gives a contribution to $\tilde{\Sigma}(k, t-t')$ equal to

$$\frac{i^2}{(2\pi)^2}\int d\mathbf{p}\int dt_1\, k_\alpha k_\beta \tilde{R}_{\alpha\beta}(\mathbf{p}, t-t')\exp[-\mu(\mathbf{k}+\mathbf{p})^2(t-t_1)]\,\theta(t-t_1)$$
$$\times(-\mu_2)(\mathbf{k}+\mathbf{p})^2\exp[-\mu(\mathbf{k}+\mathbf{p})^2(t_1-t')]\,\theta(t_1-t').$$

Picking out the coefficient of k^2 in this and the other diagrams, we obtain the series for $\mu-\sigma$:

$$\lambda^2\left\{-\mu_2-\frac{1}{3\pi^2}\int_0^\infty d\tau\int_0^\infty d\mathbf{p}\,p^2\exp(-\mu p^2\tau)\,\tilde{R}(p,\tau)\right\}$$
$$+\lambda^4\left\{-\mu_4+\frac{\mu_2}{3\pi^2}\int_0^\infty d\tau\int_0^\infty d\mathbf{p}\,p^4\tau\exp(-\mu p^2\tau)\tilde{R}(p,\tau)\right.$$
$$-\frac{1}{3(2\pi)^6}\int\limits_{\infty>t>t_1>t_2>0} dt\int dt_1\int dt_2\int d\mathbf{p}\int d\mathbf{q}(\mathbf{p}\cdot\mathbf{q})\left[1-\frac{(\mathbf{p}\cdot\mathbf{q})^2}{p^2q^2}\right]\tilde{R}(p,t-t_2)$$
$$+\tilde{R}(q,t_1)\exp[-\mu p^2(t-t_1)]\exp[-\mu(\mathbf{p}+\mathbf{q})^2(t_1-t_2)]\exp[-\mu q^2 t_2]$$
$$+\frac{2}{3(2\pi)^6}\int\limits_{\infty>t>t_1>t_2>0} dt\int dt_1\int dt_2\int d\mathbf{p}\int d\mathbf{q}\left[p^2-\frac{(\mathbf{p}\cdot\mathbf{q})^2}{q^2}\right]\tilde{R}(p,t)\,\tilde{R}(q,t_1-t_2)\exp[-\mu p^2(t-t_1)]$$
$$\left.\times\exp[-\mu(\mathbf{p}+\mathbf{q})^2(t_1-t_2)]\exp(-\mu p^2 t_2)\right\}+\text{etc.}$$

The choice of the quantities $\mu_2, \mu_4, \ldots$ is so far arbitrary except for the condition

$$\mu+\mu_2+\mu_4+\ldots = \kappa.$$

If all the μ_n are chosen as zero then $\mu = \kappa$ and the series reduces to the 'bare' perturbation series. Alternatively the μ_n may be so chosen that all the terms in the series for $\mu-\sigma$ vanish. The μ_n are then functions of $\mu = \sigma$ and σ is determined implicitly by the equation

$$\sigma+\mu_2(\sigma)+\mu_4(\sigma)+\ldots = \kappa.$$

This is the renormalized series which we shall use. It is not clear that this procedure gives the most rapidly converging sequence of approximations; however we shall not pursue this question further here but simply calculate successive approximations for σ by truncating the series at μ_2, μ_4 and μ_6.

3. Calculation of σ

We shall choose the correlation function R so as to make the evaluation of the higher-order terms in our series as simple as possible. A separable form is taken for $\tilde{R}$:

$$\tilde{R}(k,t) = (2\pi^2/k^2)\,E(k)\,D(t),$$

where $E(k)$ gives the energy spectrum of the fluid turbulence. The choice

$$E(k) = \tfrac{3}{2}v_0^2\,\delta(k-k_0)$$

is particularly convenient and has the further advantage that it is one of the spectral functions used in Kraichnan's computer simulations, so providing a check on the σ value for $\kappa = 0$.

The time dependence $D(t) = \exp(-\tfrac{1}{2}v_0^2 k_0^2 t^2)$ used by Kraichnan does not lend itself readily to the calculation of higher-order terms so we shall take instead

$$D(t) = \exp(-\Omega|t|).$$

The two cases considered are: (i) a frozen field given by $\Omega = 0$, (ii) a time-dependent field with $\Omega = dk_0 v_0$, where d has the value 0·57835.

This value of d is chosen since, in the lowest-order approximation, it gives the same value for σ as that obtained using Kraichnan's function $D(t)$. If the effective diffusivity does not depend too sensitively on the form of D then the higher-order approximations should not differ appreciably for the two forms of D.

Up to fourth order the integrals can be calculated analytically and we find

$$\mu_2 = \frac{-v_0^2}{\Omega+\sigma k_0^2},$$

$$\mu_4 = \frac{-v_0^4 k_0^2}{(\Omega+\sigma k_0^2)^3}+\frac{3}{16}\frac{v_0^4}{\sigma(\Omega+\sigma k_0^2)^2}H\left(1+\frac{\Omega}{\sigma k_0^2}\right),$$

where $H(\xi)$ denotes the function

$$-\tfrac{4}{3}+4\xi+2\xi^2-(\xi+2)(\xi^2-1)\ln[(\xi+1)/(\xi-1)].$$

At sixth order there are nineteen diagrams in all, thirteen of which can be calculated analytically. The rest were collected together into one multiple integral which was

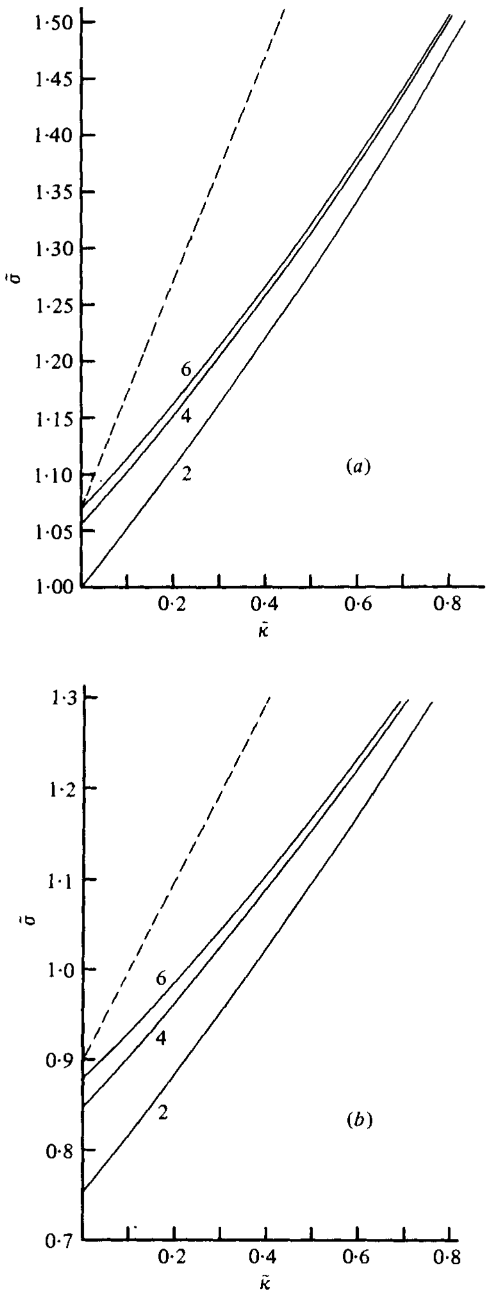

FIGURE 7. Dimensionless plot of effective long-time diffusivity against molecular diffusivity for (a) the time-independent and (b) the time-dependent case. The numbers 2, 4 and 6 denote the second-, fourth- and sixth-order approximations respectively. The broken line shows the values which would be obtained if turbulent and molecular diffusion acted independently.

analytically reduced to a double integral and evaluated numerically. For more realistic correlation functions the evaluation of the terms of the series would involve more computation but could still be carried out, at least up to fourth order.

The results are shown graphically in figure 7 in terms of the dimensionless quantities

$$\tilde{\sigma} = \sigma k_0/v_0, \quad \tilde{\kappa} = \kappa k_0/v_0.$$

The broken line shows the graph which is obtained on the assumption that the molecular and turbulent diffusivities are simply additive. It is seen that the successive approximations appear to be converging fairly rapidly especially for the frozen-field case. This is useful since the simple approximations previously given for the case $\kappa = 0$ are less accurate for the frozen field. The convergence is also quicker for larger values of κ as might be expected. The intercepts at $\kappa = 0$ for the frozen-field case are 1, 1·055 and 1·069 respectively for the second-, fourth- and sixth-order approximations. Kraichnan's computer simulations give a value of 1·1 with a possible error of a few per cent. For the time-dependent case the values obtained are 0·752, 0·847 and 0·880 and Kraichnan's calculations (which, it will be recalled, use a different time dependence) give approximately 0·9. The simple approximation mentioned in the introduction (Phythian 1975) gives 0·91.

It is interesting to note that the successive approximations are increasing and seem to approach the true value from below, the terms μ_2, μ_4 and μ_6 all being negative. We have also examined the second- and fourth-order approximations for a frozen field with a different spectral function $E(k) \propto k^4 \exp(-2k^2/k_0^2)$ ($E_2(k)$ in Kraichnan's notation) and found similar behaviour, the values for $\kappa = 0$ being 1·15 and 1·28 compared with an 'exact' value of about 1·3 from Kraichnan's (1970*a*) numerical simulation.

A preliminary investigation of the method applied to the same problem in two dimensions is interesting. For the frozen field the second-order approximation gives the same result as is obtained in three dimensions, however μ_4 is now positive and the fourth-order equation for σ has no real roots. This breakdown of the method is presumably due to the trapping of fluid particles into closed orbits in the manner described by Kraichnan (1970*a*). This would render the assumed asymptotic form of G incorrect.

4. Conclusion

The method described seems to provide successive approximations which converge fairly quickly and appear to be consistent with what is already known from Kraichnan's results for $\kappa = 0$. It fully supports Saffman's view of the destructive interference of the molecular and turbulent diffusion processes. At the same time it provides a further test of the general philosophy of renormalized perturbation expansions which underlies much recent work in turbulence theory and related fields.

It is not difficult to think of other problems which might be amenable to this approach. One possibility is the diffusion of a weak magnetic field in a conducting fluid in random motion, where it is known that second-order theories such as direct interaction may, in some circumstances, be inadequate (Kraichnan 1976*a*, *b*).

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